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The Zeeman effect for the 'anisotropic hydrogen atom' in the complex WKB approximation: II. Quantization of two-dimensional Lagrangian tori (with focal points) for the Pauli operator with spin–orbit interaction

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Abstract. On the basis of the complex WKB method, new spectral series for perturbed states of an 'anisotropic' hydrogen atom, taking into account the electron's spin polarization in a homogeneous magnetic field are constructed. These series correspond to conditionally periodic motions of a classical electron in the plane orthogonal to the field direction. The semiclassical wavefunctions are regular in the entire configuration space including the focal points and have the 'superscar' properties near the domain of classical motion, i.e. near the projection of stable two-dimensional Lagrangian tori on the configuration space. Both for Schrödinger and Pauli operators the analytical description of the semiclassical wavefunctions 'superscar' structure is obtained. This structure is non-uniform with respect to the space coordinates and has singularities with respect to the parameter $\hbar$, $\hbar \to 0$, near focal points. In the limit case of strong magnetic fields explicit analytical formulae for the semiclassical energy levels are obtained.

1. Statement of the problem

We consider the following spectral problem for the Pauli operator with spin–orbit interaction [1]

$$\hat{H}_p \Psi = E\Psi \qquad \Psi \in L_2(\mathbb{R}^3_{x,y,z}) \times \mathbb{C}^2 \qquad (\Psi = (\Psi_1, \Psi_2)') \tag{1.1}$$

where the quantum Hamiltonian has the form

$$\hat{H}_p = \hat{H}_s + \hat{V}_{s-f}$$

$$\hat{H}_s = \frac{1}{2m}\left(-i\hbar\nabla - \frac{e}{c}A\right)^2 + V_\gamma(x,y,z) \qquad V_\gamma = -\frac{e^2}{\sqrt{x^2+y^2+\gamma z^2}} \qquad \gamma > 0$$

$$\hat{V}_{s-f} = \frac{e_0\hbar}{2mc}\langle \sigma, H\rangle - \frac{e_0\hbar}{4m^2c^2}\left\langle \sigma, \left(-i\hbar\nabla - \frac{e}{c}A\right) \times E\right\rangle \tag{1.2}$$

γ is the parameter of anisotropy of the scalar potential $V_\gamma(q)$, $q = (x, y, z)$, $e_0 = -e$, e is the charge of the electron, $A = \frac{1}{2}H_0(-y, x, 0)$ is the vector potential of homogeneous magnetic field, $\hat{V}_{s-f}$ is the operator of interaction between the spin and the external field, $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices, $eE = -\nabla V_\gamma(q)$, $H = \nabla \times A$. For $\gamma = 1$ problem (1.1)–(1.2) is the *anomalous Zeeman effect* problem [1].

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The purpose of this paper is to obtain information on individual eigenvalues and eigenfunctions (spectral series) of problem (1.1)–(1.2) in the semiclassical approximation (as $\hbar \to 0$) in the widest range of classical parameters γ and H_0, including the intermediate domain of magnetic field values (10^8–10^{10} G). This is the most difficult case for study by other methods, such as regular perturbation theory and its various modifications [2–6], because in this domain, as $\gamma \sim 1$, the magnetic and Coulomb interactions are of the same order. The semiclassical approximation based on the standard multidimensional EBK–WKB–Maslov method cannot be applied here, because the corresponding three-dimensional classical system

$$\dot{p} = -\nabla_q H_s(p,q) \qquad \dot{q} = \nabla_p H_s(p,q)$$
$$(p = (p_x, p_y, p_z) \in \mathbb{R}^3_p \; q = (x, y, z) \in \mathbb{R}^3_q) \tag{1.3}$$

with Hamiltonian

$$H = H_s(p,q) = \frac{1}{2m}\left(p - \frac{e}{c}A\right)^2 + V_\gamma(q). \tag{1.4}$$

is only 'partially' integrable. Due to the axial symmetry of the electromagnetic field, system (1.3)–(1.4) has only one motion integral additional to the energy. This integral is the projection $I = xp_y - yp_z$ of the orbital momentum onto the axis z.

Nevertheless this non integrable system (with one cyclic variable) permits special families of invariant (with respect to Hamiltonian dynamics (1.3)) Lagrangian tori Λ^k, their dimension k being less than the dimension of configuration space $\mathbb{R}^3_q$.

The existence of such families in the phase space of the classical system is a result of its symmetries; the continuous axial symmetry generates the family of closed phase orbits $\Lambda^1(I)$ (cf part I) corresponding to the stationary rotation or relative equilibria of the system on the reduced phase space [7]; the discrete symmetry of Hamiltonian (1.4) with respect to the canonical change of variables $z \mapsto -z$, $p_z \mapsto -p_z$ generates the family $\Lambda^2(I, E)$, of two-dimensional isotropic tori, lying in the vicinity of $\Lambda^1(I)$ (see figure 1); E is the energy of the system. The projections $D^1_x(I)$ and $D^2_x(I, E)$ of these tori onto the configuration space lie in the plane $z = 0$, orthogonal to the magnetic field $\boldsymbol{H}$.

When invariant families of isotropic tori Λ^k, of less than full dimension ($k < n$), exist in the phase space of a classical n-dimensional system in the region of its regular (non chaotic) motion, we can carry out the basic idea of semiclassical quantization, i.e. to establish a correspondence between quantum and classical objects. In other words, we can associate this family of invariant tori with semiclassical spectral series (quasimodes)—sequences of asymptotic as $\hbar \to 0$ eigenvalues and eigenfunctions of the quantum Hamiltonian. A mathematically rigorous variant of semiclassical quantization of isotropic tori (small dimensional Lagrangian) is the *complex WKB method*. It is based on Maslov's *complex germ theory* [8, 9] and was developed in [10–12] both for scalar and matrix Hamiltonians. A complex germ is a geometric object that determines (together with the isotropic torus) the complex part of the wavefunction phase. The existence conditions of this object are similar to the conditions of orbital stability (in the linear approximation) for isotropic tori. The complex germ construction is connected with special complex solutions of a linear Hamiltonian system (*system in variations*). This system is the linearization of the initial Hamiltonian system in the vicinity of the invariant isotropic torus.

In part I [13] we quantized a family of closed phase orbits $\Lambda^1(I)$ by this method. Here we construct spectral series associated with the family of two-dimensional isotropic tori $\Lambda^2(I, E)$ for the same quantum problem (1.1)–(1.2). This is a non-trivial problem because these tori have focal points (which form caustics; see figure 1) in the radial variable. Hence

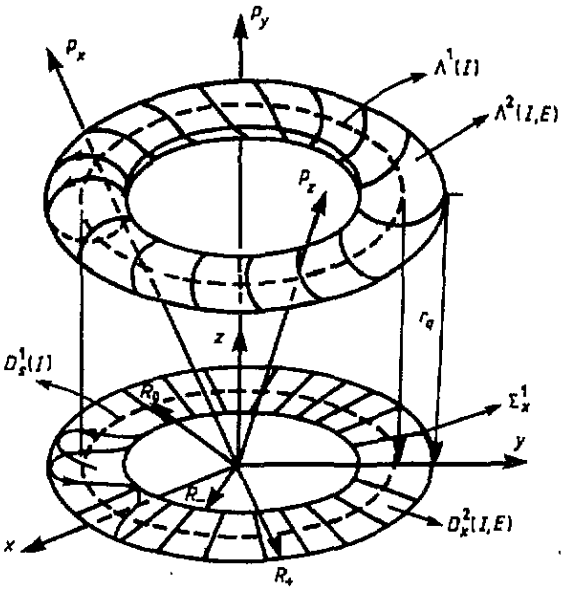


Figure 1. Quantized invariant objects in phase space $\mathbb{R}^6_{p,x}$: $\Lambda^1(I)$, closed phase curve; $\Lambda^2(I, E)$, two-dimensional Lagrangian (isotropic) torus; $R_\pm$, radial turning points; $\Sigma_x^1 = S_+^1 \cup S_-^1$, caustics; $D_x^2(I, E)$, projection of $\Lambda^2(I, E)$ on configuration space $\mathbb{R}^3_x$ in the plane $z = 0$.

(as in the full dimensional case ($k = n$) in the real WKB method), singularities of the wavefunction arise in the vicinity of such points.

The main result of this part of the work is as follows. In sections 2–6 for problem (1.1)–(1.2) we construct new semiclassical spectral series, i.e. sequences of asymptotic as $\hbar \to 0$ eigenfunctions $\Psi_{n,\zeta}(q, \hbar)$ and eigenvalues $E_{n,\zeta}(\hbar)$, corresponding to the conditionally periodic two-frequency motion of the classical electron in the plane $z = 0$, orthogonal to the field $\boldsymbol{H}$. Here $n = (n_1, n_2, n_3)$, n_j are integer numbers, $\zeta = \pm 1$ is the electron spin. The wavefunctions obtained are regular in the entire configuration space (including focal points) and have the 'superscarring' property (cf [13]), i.e. there is an accumulation of density in the coordinate space near the two-dimensional domain $D_x^2(I, E)$ of the classical motion of the electron in the plane $z = 0$. In section 6, using the general formulae from [10–12], the structure of superscars of semiclassical wavefunctions, associated with the *family of stable tori* $\Lambda^2(I, E)$ is investigated. This structure is non uniform with respect to the space coordinates and possesses singularities with respect to the parameter $\hbar$, $\hbar \to 0$, near the focal points. Let us remark that for *unstable* periodic orbits a semiclassical theory of the *scar* phenomenon for semiclassical wavefunctions (which was discovered by Heller [14]) has been constructed in configuration space by Bogomolny [15] and in phase space by Berry [16] (see also Robnik [17]).

In section 7 explicit analytic formulae for semiclassical energy levels in the limit case of strong ($H_0 \to \infty$) magnetic fields and for any value of the anisotropy parameter $\gamma > 0$ are presented. We obtain these formulae in the form of a regular expansion in the parameter $\varepsilon_s = \varepsilon^{1/2} \sim H_0^{-1/2} \ll 1$. Here $\varepsilon = a_H/a_B$ as the ratio of the Larmor radius $a_H = (ch/e_0H_0)^{1/2}$ to the electron's Bohr radius $a_B = \hbar^2/me_0^2$.

The possibility of carrying out this investigation is due to the following fact. From the analytical point of view the complex WKB method reduces the construction of the germ

asymptotics for the quantum problem (1.1)–(1.2) to the integration of three systems of ordinary differential equations: the nonlinear Hamiltonian system (1.3) (to find isotropic tori $\Lambda^2(I, E)$), the corresponding linear Hamiltonian system in variations (to construct the complex germ) and a linear system—the polarization equation—to describe the electron's spin evolution.

Hence, to analyse the semiclassical energy level dependence on the other classical parameters of the system, we can use well developed asymptotic methods. In particular, to study the semiclassical energy levels in limit cases of the magnetic field value H_0, we use the Poincaré–Linstedt method [18] for the nonlinear equations of motion, the averaging method for linear Hamilton systems with high frequency parametric perturbation [19], for the equation in variations, and the standard perturbation theory for the electron's spin polarization equation.

As is known ([20–24]), for spectral problems with matrix Hamiltonians such as (1.1)–(1.2), the orbital ($q = (x, y, z)$) and spin ($\zeta = \pm 1$) variables can be separated up to $O(\hbar^2)$, $\hbar \to 0$ in the WKB approximation. The same idea is valid for the complex WKB method [12]. The semiclassical spectral series of the initial spectral problem have the following structure (outside the neighbourhood of focal points)

$$\Psi_{n,\zeta}(q, \hbar) = \Psi_n(q, \hbar) f_\zeta(q) \qquad E_{n,\zeta}(\hbar) = E_n(\hbar) + \hbar\mu_\zeta + O(\hbar^2) \tag{1.5}$$

($E_n(\hbar)$, $\Psi_n(q, \hbar)$ is the semiclassical spectral series of the scalar Schrödinger operator $\hat{H}_s$)

$$\hat{H}_s \Psi_E(q, \hbar) = E\Psi_E(q, \hbar) \qquad \Psi_E(q, \hbar) \in L_2(\mathbb{R}^3_q) \tag{1.6}$$

$$\hat{H}_s = \frac{1}{2m}\left(-i\hbar\nabla - \frac{e}{2c}[H, q]\right)^2 - \frac{e^2}{\sqrt{x^2 + y^2 + \gamma z^2}} \tag{1.7}$$

(E is the spectral parameter), $q = (x, y, z)$ and μ_ζ, $f_\zeta(q)$ is a solution of the spectral problem for the polarization equation on the invariant torus $\Lambda^2(I, E) = \{q = Q(\tau_1, \tau_2), p = P(\tau_1, \tau_2), 0 \leqslant \tau_j < 2\pi\}$, τ_j being the angular coordinates on the torus;

$$-i\frac{d}{dt} f_\zeta + \Pi|_{\Lambda^k} f_\zeta = \mu_\zeta f_\zeta \qquad f_\zeta : \Lambda^k \to \mathbb{C}^2. \tag{1.8}$$

Here $f_\zeta = f_\zeta(q = Q(\tau_1, \tau_2))$ and d/dt is the differentiation along the trajectories of system (1.3) and the polarization matrix Π has the form

$$\Pi = \frac{e_0}{2mc}\left\langle \sigma, H(q) - \frac{1}{2mc}[\dot{q} \times E]\right\rangle. \tag{1.9}$$

In the case of the Dirac operator a similar reduction of the initial spectral problem to solve a scalar problem of type (1.6) and the corresponding polarization equation for the electron spin was performed in [25, 26]. For problem (1.1)–(1.2) obviously, matrix (1.9) is the limit of the corresponding polarization matrix in [26] (see formula (5.132)) as $\beta = v/c \to 0$, where v is the speed of the electron.

Thus, we have to solve two auxiliary problems. The first of them (the spectral problem for the Schrödinger equation) has an original scientific meaning by itself.

2. Construction of the family of two-dimensional isotropic tori $\Lambda^2(I, E)$ with the complex germ $r^3(\Lambda^2(I, E))$

In cylindrical coordinates ρ, z, φ the Hamiltonian function H_s corresponding to operator (1.7) has the form

$$H_s = H_s(p, p_\varphi, q) = \frac{p_\rho^2}{2m} + \frac{p_z^2}{2m} + \frac{p_\varphi^2}{2m\rho^2} + \tfrac{1}{8}m\omega_H^2\rho^2 - \frac{e_0^2}{\sqrt{\rho^2 + \gamma z^2}} + \tfrac{1}{2}\omega_H p_\varphi. \tag{2.1}$$

The variables p_ρ, p_z and p_φ are canonically conjugate to the variables ρ, z, $\varphi(\mathrm{mod}\,2\pi)$; $\omega_H = e_0 H_0/mc$ is the cyclotron frequency, $e = -e_0$, $e_0 > 0$ is the charge and m is the mass of the electron.

In a Hamiltonian system with a cyclic (angular) variable one can always single out a special family of closed trajectories $\Lambda^1(I)$—circles, which are stationary motions or relative equilibrium states of the system in reduced phase space with coordinates ρ, z, p_ρ, p_z. As shown in part I ([13]), the explicit expression for $\Lambda^1(I)$ can be given when the values of the energy interval E and the momentum projection integral I ($I = p_\varphi$) are related by the following formula

$$E = \alpha(I) = \omega_H I/2 - e_0^2/(2R_0(I)) + m\omega_H^2 R_0^2(I)/4. \tag{2.2}$$

Here $R_0(I)$ is a critical point of the 'effective' potential

$$V_I(\rho, z) = I^2/(2m\rho^2) + m\omega_H^2\rho^2/8 - e_0^2(\rho^2 + \gamma z^2)^{-1/2} \tag{2.3}$$

in the plane $z = 0$. $R_0(I)$ is the solution of the equation

$$m^2\omega_H^2 R_0^4/4 + e_0^2 m R_0 = I^2. \tag{2.4}$$

Let us fix the values of the motion integrals of energy E, $E = E_0$ and of momentum p_φ, $p_\varphi = I_0$, so that the point (I_0, E_0) does not lie on the bifurcation curve $\Sigma^1 = \{(I, E), E = \alpha(I)\}$ (see [7] and [13]). Let us also consider the intersection of the coordinate plane $\Pi_z = \{(p, q), p_z = 0, z = 0\}$ with the level $M_{E_0} = \{(p, q), H_{I_0}(p, q) = E_0\}$ of the reduced Hamiltonian function

$$H_{I_0}(p, q) = H(p, I_0, q) = p_\rho^2/2m + 2mp_z^2/2m + V_{I_0}(\rho, z) + \omega_H I_0/2$$

in the reduced phase space $\mathbb{R}_p^2 \times \mathbb{R}_q^2$.

The intersection $M_{E_0} \cap \Pi_z$ defines the closed curve

$$\bar{\Lambda}^1(I_0, E_0) = \{(p, q), p_\rho^2/2m + V_{I_0}(\rho, 0) + \omega_H I_0/2 = E_0, p_z = z = 0\}. \tag{2.5}$$

The product

$$\bar{\Lambda}^1(I_0, E_0) \times S^1(I_0) = \Lambda^2(I_0, E_0) \tag{2.6}$$

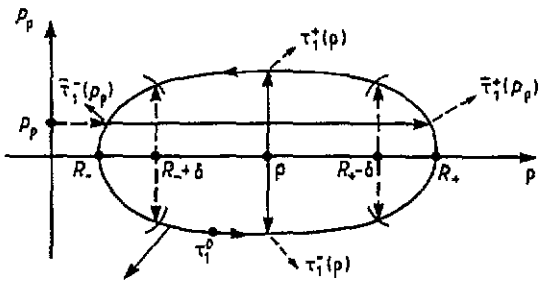
in which $S^1(I_0) = \{(p_\varphi, \varphi), p_\varphi = I_0, \varphi \in [0, 2\pi)\}$ is a circle (see figure 2) gives a two-dimensional Lagrangian torus $\Lambda^2(I_0, E_0)$ in the initial phase space $\mathbb{R}_p^3 \times \mathbb{R}_q^3$. The torus $\Lambda^2(I_0, E_0)$ is invariant with respect to the phase flow g_H^t of the Hamiltonian system (1.3)–(1.4). This fact follows from the integrability of the Hamiltonian system (1.3) in the plane Π_z and from the relations $\partial H/\partial p_z|_{\Lambda^2} = \partial H/\partial z|_{\Lambda^2} = 0$.

The motion of the Hamiltonian system (1.3) along the torus $\Lambda^2(I_0, E_0)$ is conditionally periodic with frequencies

$$\begin{aligned}
&\omega_1(I_0, E_0) = 2\pi T_1^{-1}(I_0, E_0)\\
&T_1(I_0, E_0) = 2\int_{R_-(I_0,E_0)}^{R_+(I_0,E_0)} \left(\frac{2}{m}\left(E_0 - \frac{\omega_H I_0}{2} - V_{I_0}(\rho, 0)\right)\right)^{-1/2} \mathrm{d}\rho\\
&\omega_2(I_0, E_0) = T_1^{-1}(I_0, E_0)\int_0^{T_1(I_0,E_0)} \frac{\partial V_I}{\partial I}(X(\omega_1 t + \tau_1, E_0, I_0)\,\mathrm{d}t
\end{aligned} \tag{2.7}$$

where $R_\pm(I_0, E_0)$ are the radial focal points of the effective potential (2.3) with $z = 0$: $V_{I_0}(\rho, 0) = E_0 - \omega_H I_0/2$; $X(\omega_1 t + \tau_1, E_0, I_0) = R(t)$ is the solution of the 'radial' Newton equation

$$m\ddot{R} + \partial V_{I_0}(R, 0)/\partial R = 0 \tag{2.8}$$



$$\bar\Lambda^1(I,E):\ \frac{P^2(\tau_1)}{2m}+V_I(X(\tau_1),0)=E-\frac{\omega_H I}{2}\qquad 0\leqslant\tau_1<2\pi$$
$$\tau_1^\pm(\rho):\ \rho=X(\tau_1)\qquad R_-+\delta\leqslant\rho\leqslant R_+-\delta\qquad \delta<1$$
$$\bar\tau_1^\pm(p_\rho):\ p_\rho=P(\tau_1)\qquad \tau_1^\pm-\varepsilon(\delta)\leqslant\tau_1\leqslant\tau_1^\pm+\varepsilon(\delta)$$

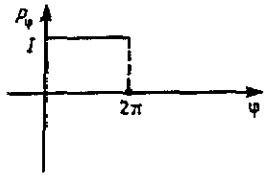


Figure 2. Projection of the family of invariant two-dimensional Lagrangian tori $\Lambda^2(I,E)=\bar\Lambda^1(I)\times S^1(I)$ in the plane $z=p_z=0$; $R_\pm$ are the radial turning points.

so that in coordinates of the phase space $\mathbb{R}^6_{p,q}$ the periodicity condition for the motion on the torus Λ^2 (2.6) has the form†

$$p=P_t=(p_\rho=P(\omega_1 t+\tau_1)=m\omega_1 X'(\omega_1 t+\tau_1,E_0,I_0),\ p_\varphi=I_0,\ p_z=0)^t$$
$$q=Qt=(\rho=X(\omega_1 t+\tau_1,E_0,I_0),\ \varphi=\omega_2 t+\tau_2+\theta(\omega_1 t+\tau_1),\ z=0)^t \tag{2.9}$$

where $\tau=(\tau_1,\tau_2)$, $\tau_j\in[0;2\pi[$ are angular coordinates on Λ^2, $\theta(\tau_1+2\pi)=\theta(\tau_1)$. Here and below the derivative with respect to τ_1 is denoted by prime.

The projection D_x^2 of the torus $\Lambda^2(I_0,E_0)$ on configuration space, i.e. the domain of 'light' or 'bright region', is an annulus lying in the plane $z=0$ (see figure 1)

$$D_x^2=\{(\rho,z,\varphi),\ \rho\in[R_-(I_0,E_0),R_+(I_0,E_0)],\ z=0,\ \varphi\in[0,2\pi)\}.$$

The boundary of D_x^2 is formed by two circles S_-^1 and S_+^1 with radii $R_-(I_0,E_0)$ and $R_+(I_0,E_0)$ respectively. This boundary is a caustic curve. It is the envelope of the family of 'rays', i.e. of the projections of the trajectories of Hamiltonian system (1.3), that fill the torus $\Lambda^2(I_0,E_0)$ everywhere densely in configuration space. We naturally assume the invariant Lagrangian torus $\Lambda^2(I_0,E_0)$ (2.6) to be non resonant (i.e. on the ring of integers the frequencies $\omega_1(I_0,E_0)$ and $\omega_2(I_0,E_0)$ (2.7) are linearly independent). We also assume this property to be valid in a certain neighbourhood $U_{I_0,E_0}\subset\mathbb{R}^1_I\times\mathbb{R}^1_E$ of the point $(I_0,E_0)\notin\Sigma^1$ (see e.g. [7]).

Now we construct the complex germ $r^3(\Lambda^2(I,E))$ on the family of two-dimensional Lagrangian tori $\Lambda^2(I,E)$. Consider the system in variations associated with (1.3) in the vicinity of Λ^2 given in (2.6). This system has the following form

$$\dot a=\mathcal{H}^{\mathrm{var}}|_{\Lambda^2}a\qquad a=(w,z)^t\ w=\delta p\in\mathbb{C}^3\ z=\delta q\in\mathbb{C}^3 \tag{2.10}$$

† For the sake of simplicity we shall omit the dependence on the parameters E and I, wherever it does not cause confusion.

where the matrix in variations $\mathcal{H}^{\text{var}}$ is

$$\mathcal{H}^{\text{var}} = J\frac{\partial^2 H}{\partial \eta^2} = \begin{pmatrix} -H_{qp} & -H_{qq} \\ H_{pp} & H_{pq} \end{pmatrix}_{6\times 6}$$

$$\eta = (p, q) \in \mathbb{R}^6_{p,q}\ p = (p_\rho, p_\varphi, p_z)\ q = (\rho, \varphi, z). \tag{2.11}$$

Here $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{6\times 6}$, $(1 = ((\delta_{ij}))_{3\times 3})$, is the standard symplectic matrix [7]. Matrix $\mathcal{H}^{\text{var}}$ is calculated at the points of conditionally periodic motion (2.9) on the invariant tori Λ^2 (2.6). Due to the axial symmetry, matrix (2.11) depends on the radial motion $R(t)$ only. Hence (2.10) is a system with T_1 periodic coefficients.

For system (2.10) we find three linearly independent solutions $a_k(t, \tau)$, $k = 1, 2, 3$, $\tau = (\tau_1, \tau_2) \in \Lambda^2$, satisfying the following conditions:

(i) $a_1(t, \tau)$ and $a_2(t, \tau)$ form a basis on the plane tangent to Λ^2 at the torus points (P_t, Q_t) (2.9) and have the form

$$a_1(t, \tau) = \left(\frac{\partial P_t}{\partial \tau_1}, \frac{\partial Q_t}{\partial \tau_1}\right)^t = (P'(\omega_1 t + \tau_1), 0, 0; X'(\omega_1 t + \tau_1), \theta'(\omega_1 t + \tau_1), 0)^t$$

$$a_2(t, \tau) = (\partial P_t/\partial \tau_2, \partial Q_t/\partial \tau_2)^t = (0, 0, 0; 0, 1, 0)^t \tag{2.12}$$

(ii) all three solutions are skew-orthogonal to each other

$$\{a_j, a_k\}(t, \tau) = \langle a_j, Ja_k\rangle_{\mathbb{R}^6_z} = 0 \qquad j, k = 1, 2, 3 \tag{2.13}$$

(iii) the solution $a_3(t, \tau)$ is complex and satisfies the dissipativity condition [9, 12]

$$\{a_3, \bar{a}_3\}(t, \tau) = 2\mathrm{i} \tag{2.14}$$

(here and below the bar over a vector means complex conjugation). It follows from (2.12)–(2.14) that the vector $a_3(t, \tau)$ can have the following form

$$a_3(t, \tau) = (0, 0, \omega(t, \tau), 0, 0, \chi(t, \tau))^t. \tag{2.15}$$

Then it is easy to verify that $a_3(t, \tau)$ (2.15) is a solution of the system in variations (2.10), when $w = m\dot{\chi}$, where χ is a complex solution of the following Hill equation (the equation of an oscillator with variable periodic frequency)

$$\ddot{\chi} + \Omega(t)\chi = 0 \qquad \Omega(t + T_1) = \Omega(t) \tag{2.16}$$

$$\Omega(t) = \frac{1}{2m}\frac{\partial V_I(\rho, z^2)}{\partial z^2}\bigg|_{z=0, \rho=R(t)} = \frac{\gamma e_0^2}{mR^3(t)} = \frac{\gamma e_0}{mX^3(\omega_1 t + \tau_1, I, E)} = \tilde{\Omega}(\omega_1 t + \tau_1, I, E).$$

Here $R(t) = X(\omega_1 t + \tau_1, I, E)$ is the periodic solution of the Newton equation (2.8), V_I is the effective potential (2.3) and $\tilde{\Omega}(\xi, I, E)$ is a 2π-periodic function of ξ.

As was mentioned above, the existence of the complex germ $r^n(\Lambda^k)$ is equivalent to the condition of orbital stability of the isotropic torus Λ^k. For a two-dimensional isotropic torus Λ^2 (2.6), lying in the four-dimensional plane $z = p_z = 0$ of the classical phase space $\mathbb{R}^6_{p,q}$, this condition means that the torus is stable with respect to the small deviations $\delta p_z = w$, $\delta z = \chi$ transversal to $\Lambda^2(I, E)$. Hence *we require all the solutions of the Hill equation (2.16) to be bounded for* $t \in (-\infty, +\infty)$. For a fixed value of the magnetic field H_0, let us denote the domain of stability of (2.16) by $V(H_0) \in U_{(I_0, E_0)}$. Then according to Floquet theory (e.g. [18, 19]) a basis of solutions of this equation consists of functions χ and $\bar{\chi}$ such that

$$\chi(t + T_1, I, E) = \mathrm{e}^{\mathrm{i}\beta(I,E)}\chi(t, I, E) \qquad t \in \mathbb{R}^1 \tag{2.17}$$

where the characteristic exponent $\beta(I, E)$ is real and does not depend on the point on the torus $\Lambda^2(I, E)$. Normalizing the solution χ by the condition $m(\dot{\chi}\bar{\chi} - \dot{\bar{\chi}}\chi) = 2\mathrm{i}$ we obtain the dissipativity condition (2.14) for the vector $a_3(t, \tau)$.

Note. Due to the explicit form of the variable frequency $\Omega(t)$, the stable solution (2.17) of equation (2.16), can be represented in the following form: $\chi(t) = Z(\omega_1 t + \tau_1, I, E)$. Obviously here $Z(\tau_1, I, E)$ is a stable solution of the equation

$$\omega_1^2 \, \mathrm{d}^2 Z/\mathrm{d}\tau_1^2 + \bar{\Omega}(\tau_1) Z = 0 \tag{2.18}$$

where $\bar{\Omega}(\tau_1 + 2\pi) = \bar{\Omega}(\tau_1)$, $Z(\tau_1 + 2\pi, I, E) = \exp\{\mathrm{i}\beta(I, E)\} Z(\tau_1, I, E)$.

At any point $\tau = (\tau_1, \tau_2) \in \Lambda^2$ we consider the three-dimensional complex plane $r^3(\tau) = \{\sum_{j=1}^3 r_j(\tau)\alpha_j, \alpha_j \in \mathbb{C}\}$, where $r_j(\tau) = a_j(t, \tau)|_{t=0}$, $j = 1, 2, 3$. According to (2.12)–(2.14) this plane is Lagrangian, dissipative and contains the plane tangent to Λ^2 at the point τ. The family of planes $r^3(\tau)$, $\tau \in \Lambda^2$ defines the complex germ $r^3(\Lambda^2(I, E))$ on the invariant two-dimensional isotropic torus $\Lambda^2(I, E)$ for $(I, E) \in V(H_0)$.

The pair $[\Lambda^2(I, E), r^3(\Lambda^2(I, E))]$ plays a key role in the construction of the semiclassical spectral series for problem (1.1)–(1.2) by the complex WKB method.

3. Quantization conditions for the family of Lagrangian tori $\Lambda^2(I, E)$ with the complex germ $r^3(\Lambda^2(I, E))$

General quantization conditions of Bohr–Sommerfeld type for families of isotropic tori with complex germ were obtained in [10–12]. Since the dimension of these tori is rigorously less than the dimension of configuration space, instead of the Maslov index for tori of full dimension, a new topological characteristic appears in the quantization conditions. These characteristics are the indices $\hat{\beta}_j$ of the complex germ along the basis cycles of the isotropic tori. Let us calculate these characteristics in the considered case for two basis cycles, $\gamma_\rho = \{\tau_2 = \text{const}, 0 \leqslant \tau_1 < 2\pi\}$ and $\gamma_\varphi = \{\tau_1 = \text{const}, 0 \leqslant \tau_2 < 2\pi\}$, of the torus $\Lambda^2(I, E)$ (see figure 1). To do this we compose 3×3 matrices $B(\tau)$ and $C(\tau)$ of 'impulse' and 'coordinate' components of the vectors $r_j(\tau) = a_j(t, \tau)|_{t=0}$, $j = 1, 2, 3$ that form a basis on the plane $r^3(\tau)$, $\tau \in \Lambda^2(I, E)$. If we recall (2.12), (2.15) and (2.18), we get

$$B(\tau) = \begin{pmatrix} P'(\tau_1) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & m\omega_1 Z'(\tau_1) \end{pmatrix} \qquad C(\tau) = \begin{pmatrix} X'(\tau_1) & \theta'(\tau_1) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & Z(\tau_1) \end{pmatrix}. \tag{3.1}$$

The 2π-periodic functions $P(\tau_1)$ and $X(\tau_1)$ determine the closed curve $\bar{\Lambda}^1(I, E)$ (see figure 2), and $Z(\tau_1)$ is a stable solution of the Hill equation (2.18). The complex germ indices $\hat{\beta}_j$ along the basis cycles γ_j, $j = 1, 2$ of the torus $\Lambda^2(I, E)(\gamma_1 = \gamma_\rho; \gamma_2 = \gamma_\varphi)$ are defined as the complete increment of the argument of the determinant of the complex non-singular matrix $B(\tau) + \mathrm{i}C(\tau)$

$$\hat{\beta}_j = \operatorname{Arg} \det(B(\tau) + \mathrm{i}C(\tau))|_{\gamma_j}. \tag{3.2}$$

Since the matrices $B(\tau)$ and $C(\tau)$ do not depend on τ_2, $\hat{\beta}_2 = \hat{\beta}_{\gamma_\varphi} = 0$. For the cycle γ_ρ, taking into account (3.1) we have

$$\hat{\beta}_1 = \hat{\beta}_{\gamma_\rho} = \operatorname{Arg}(P'(\tau_1) + \mathrm{i}X'(\tau_1))(m\omega_1 Z'(\tau_1) + \mathrm{i}Z(\tau_1))|_0^{2\pi} = 2\pi + \beta(I, E) \tag{3.3}$$

where $\beta(I, E)$ is the characteristic exponent of the Floquet solution (2.17). In (3.3) we used the inequalities $Z(\tau_1) \neq 0$, $\operatorname{Im}(Z'(\tau_1)/Z(\tau_1)) > 0$, that are implied by the normalization condition of the solution $\chi(\tau)$.

Thus the quantization conditions for the family $\Lambda^2(I, E)$ with complex germ $r^3(\Lambda^2(I, E))$ are (cf [10])

$$\frac{1}{2\pi\hbar} \oint_{\gamma_\varphi \in \Lambda^2(I,E)} p \, \mathrm{d}q = l \tag{3.4}$$

$$\frac{1}{2\pi\hbar}\oint_{\gamma_\rho\in\Lambda^2(I,E)} p\,\mathrm{d}q = n_\rho + \frac{(\beta(I,E)+2\pi)}{2\pi}(\nu+\tfrac{1}{2}) \qquad \nu = 0,1,2,\ldots \tag{3.5}$$

n_ρ being natural numbers, l integers, $l = 0, \pm 1, , \ldots, l = l(\hbar)$, $|l(\hbar)| \to \infty$, $n_\rho = n_\rho(\hbar) \to \infty$ as $\hbar \to 0$. We have chosen the basis cycles γ_φ and γ_ρ such that the motion along them implies only the change in one coordinate, φ or ρ. Condition (3.4) gives $I = lh$. The second condition gives the semiclassical energy levels $E = E_{l,n_\rho,\nu}(\hbar)$ for the Schrödinger operator (1.7).

4. Semiclassical spectral series for the Schrödinger operator

The quantization condition (3.5) can be rewritten in the following form

$$\sqrt{2m}\int_{R_-(I,E)}^{R_+(I,E)}\left(E - \frac{\omega_H I}{2} - V_l(\rho,0)\right)^{1/2}\mathrm{d}\rho|_{I=l\hbar}$$
$$= \pi\hbar\left(n_\rho + \left(\frac{\beta(l\hbar,E)}{2\pi}+1\right)(\nu+\tfrac{1}{2})\right). \tag{4.1}$$

This condition determines the spectral parameter $E + E_{l,n_\rho,\nu}(\hbar)$; here $R_\pm(I,E)$ are the radial 'turning' points, i.e., zeros of the integrand in the left-hand part of condition (4.1). If the numbers $|l(\hbar)| \to \infty$ and $n_\rho = n_\rho(\hbar) \to \infty$, $\hbar \to 0$ are related to the parameter $\hbar$ by the following conditions

$$l(\hbar)\hbar \to I_0 \qquad n_\rho(\hbar)\hbar \to (2\pi)^{-1}\oint_{\gamma_\rho\in\Lambda^2(I_0,E_0)} p\,\mathrm{d}q \qquad \hbar \to 0 \tag{4.2}$$

then *the semiclassical series of eigenvalues* $E_{l,n_\rho,\nu_2}(\hbar)$ *for the Schrödinger operator* determined by (4.1) are associated with the motion of the classical system (1.3) along the invariant torus $\Lambda^2(I_0,E_0)$ (2.6) in the limit as $\hbar \to 0$, $(I_0,E_0) \notin \Sigma^1$.

Thus for all values of H_0, our method reduces the calculation of the semiclassical energy spectra, related to the regular quasiperiodic motion of the electron in the plane $z = 0$, to solving the algebraic equation (4.1), with the values of E and I in the stability domain $U_{I_0,E_0}(H_0)$ of the Hill equation (2.16).

5. The polarization equation (spin correction)

The polarization matrix (1.9) for the family $\Lambda^2(I,E)$ is a matrix with T_1-periodic elements that can be calculated as in [26]:

$$\Pi|_{\Lambda^2(I,E)} = \frac{e}{2mc}\left(H_0 + \frac{eI}{2mcR^3(t)}\right)\hat{\sigma}_3 \qquad \hat{\sigma}_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

The spectral problem (1.8) for the two-component spinor f_ζ on $\Lambda^2(I,E)$ has the form

$$-\mathrm{i}\mathrm{d}f_\zeta/\mathrm{d}t + e(2mc)^{-1}(H_0 + eI/2mcR^3(t))\hat{\sigma}_3 f_\zeta = \mu_\zeta f_\zeta$$

where the differentiation operator $\mathrm{d}/\mathrm{d}t$ along the trajectories (2.9) has the form

$$\mathrm{d}/\mathrm{d}t|_{\Lambda^2} = \langle\dot{x},\nabla_x\rangle + \langle\dot{p},\nabla_p\rangle = \dot{\varphi}\partial/\partial\varphi + \dot{\rho}\partial/\partial\rho + \dot{p}_\rho\partial/\partial p_\rho.$$

It is convenient to pass to the global coordinate $\tau_1 \bmod 2\pi$, on the curve $\bar{\Lambda}^1(I,E)$; then due to (2.9): $\dot{\rho}\partial/\partial\rho + \dot{p}_\rho\partial/\partial p_\rho = \omega_1\partial/\partial\tau_1$. Consequently, in global coordinates (τ_1,τ_2) on $\Lambda^2(I,E)$ we obtain the spectral problem

$$-\mathrm{i}\omega_2(I,E)\partial/\partial\tau_2 - \mathrm{i}\omega_1(I,E)\partial/\partial\tau_1 + a(\tau_1)\hat{\sigma}|f_\zeta\rangle = \mu_\zeta|f_\zeta\rangle$$
$$\zeta = \pm 1 \qquad f_\zeta(\tau_1+2\pi,\tau_2) = f_\zeta(\tau_1,\tau_2) \qquad f_\zeta(\tau_1,\tau_2+2\pi) = f_\zeta(\tau_1,\tau_2). \tag{5.1}$$

Here

$$a(\tau_1) = [H_0 + e_0 I/(2mcX^3(\tau_1, E, I))]e_0/2mc. \tag{5.2}$$

We take (5.1) to a one-dimensional spectral problem on the circle through the transformation $f_\zeta(\tau_1, \tau_2) = \exp\{ik_2\tau_2\}v_\zeta(\tau_1)$, $k_2 \in \mathbb{Z}$ (separation of variables):

$$-i\omega_1 \partial v_\zeta/\partial\tau_1 + a(\tau_1)\hat{\sigma}_3 v_\zeta = (\mu_\zeta - k_2\omega_2)v_\zeta \qquad v_\zeta(\tau_1 + 2\pi) = v(\tau_1) \qquad k_2 \in \mathbb{Z}. \tag{5.3}$$

From here we get

$$\begin{aligned} &v_\zeta(\tau_1) = v_\zeta \exp\{ik_1\tau_1\}\exp\{i\zeta g(\tau_1)\} \\ &g(\tau_1) = \frac{1}{\omega_1}\left[\tau_1 \frac{1}{2\pi}\int_0^{2\pi} a(\tau_1)\,d\tau_1 - \int_0^{\tau_1} a(\tau_1)\,d\tau_1\right] \\ &\mu_\zeta = k_1\omega_1 + k_2\omega_2 + \frac{1}{2\pi}\zeta\int_0^{2\pi} a(\tau_1)\,d\tau_1 \qquad k_1, k_2 \in \mathbb{Z} \end{aligned} \tag{5.4}$$

where the vector v_ζ is constant; $v_{\pm 1} = (1, 0)^t$, $v_{-1} = (0, 1)^t$. Thus we have proved that the eigenfunctions and eigenvalues of the spectral problem (5.1) have the form

$$f_\zeta^{(k_1,k_2)}(\tau_1, \tau_2) = \exp[i(k_1\tau_1 + k_2\tau_2)]\exp\left(i\left[\frac{1}{2\pi}\tau_1\int_0^{2\pi} a(\tau_1)\,d\tau_1 - \int_0^{\tau_1} a(\tau_1)\,d\tau_1\right]\zeta/\omega_1\right)v_\zeta$$

$$\mu_\zeta^{(k_1,k_2)} = k_1\omega_1 + k_2\omega_2 + \zeta\frac{1}{2\pi}\int_0^{2\pi} a(\tau_1)\,d\tau_1 \qquad k_1, k_2 \in \mathbb{Z}\ \zeta = \pm 1. \tag{5.5}$$

Without loss of generality (see the remark in the next section) let us put $k_1 = k_2 = 0$ in (5.5). Then we obtain the following *spectral series* (E_n, Ψ_n), *associated with the family* $\Lambda^2(I, E)$ *for the Pauli operator (1.2)*:

$$\begin{aligned} E_{l,n_\rho,\nu,\zeta}(\hbar) = E_{l,n_\rho,\nu}(\hbar) + \frac{h}{2\pi}\zeta\int_0^{2\pi}\frac{e}{2mc}\left(H_0 + \frac{eI}{2mc}X^3(\tau_1, E, I)|_{I=l\hbar, E=E^t_{l,n,\rho,\nu}}\right)d\tau_1 \\ +O(\hbar^2) \end{aligned} \tag{5.6}$$

where $E_{l,n_\rho,\nu}(\hbar)$ is the *spectral series for the Schrödinger operator (1.7)*.

6. Semiclassical wavefunctions 'superscarred' by the family of two-dimensional Lagrangian tori $\Lambda^2(I, E)$ with complex germ $r^3(\Lambda^2(I, E))$

In this section we associate the family $[\Lambda^2(I, E), r^3(\Lambda^2(I, E))]$, $(I, E) \in V(H_0)$ quantized by conditions (3.4)–(3.5) with the semiclassical wavefunctions corresponding to semiclassical energy levels $E_{l,n_\rho,\nu}(\hbar)$ (4.1) (for the Schrödinger operator (1.7)) and $E_{l,n_\rho,\nu,\zeta}(\hbar)$ (5.6) (for the initial problem (1.1)–(1.2)). In both cases these functions are determined by the Maslov canonical operator with *complex phase* [9–12] on a special function ϕ on the torus $\Lambda^2(I, E)$:

(i) for the scalar problem (the Schrödinger operator)

$$\begin{aligned} &\phi(\tau_1, \tau_2) = 1 \Rightarrow \Psi_{l,n_\rho,\nu}(q, \hbar) = (\hat{K}_{[\Lambda^2(I,E),r^3(\Lambda^2(I,E))]}1)(q, \hbar) \\ &I = l\hbar \qquad E = E_{l,n_\rho,\nu}(\hbar) \end{aligned} \tag{6.1}$$

(ii) for the vector problem (the Pauli operator)

$$\begin{aligned} &\phi(\tau_1, \tau_2) = f_\zeta(\tau_1, \tau_2) = v_\zeta e^{i\zeta g(\tau_1)} \Rightarrow \Psi_{l,n_\rho,\nu,\zeta}(q, \hbar) = (\hat{K}_{[\Lambda^2, r^3]}f_\zeta(\tau_1, \tau_2))(q, \hbar) \\ &\quad = v_\zeta(\hat{K}_{[\Lambda^2(I,E),r^3(\Lambda^2(I,E))]}e^{i\zeta g(\tau_1)})(q, \hbar) \\ &I = l\hbar \qquad E = E_{l,n_\rho,\nu,\zeta}(\hbar) \qquad \zeta = \pm 1. \end{aligned} \tag{6.2}$$

The electron spin polarization vector $v_\zeta \in \mathbb{C}^2$ does not depend on the orbital coordinates $q \in \mathbb{R}^3_q$. Function g was obtained in (5.4). The properties of (6.1)–(6.2) are:

(i) functions (6.1) and (6.2) are smooth in the entire configuration space including the neighbourhoods of caustics (see section 2 and figure 1),

(ii) they satisfy the spectral problems (1.1)–(1.2) and (1.6) up to $O(\hbar^{3/2}), \hbar \to 0$. Namely, substituting (6.1) and (6.2) in (1.1) and (1.6) respectively we obtain a residue of order $\hbar^{3/2}, \hbar \to 0$ in the L_2 norm,

(iii) up to $O(\hbar^{1/2}), \hbar \to 0$ these functions form an asymptotic complete orthonormal set of states (the index + means complex conjugation)

$$\int d^3q\sqrt{g}\Psi_n^+\Psi_{n'} = \delta_{nn'} + O(\hbar^{1/2}) \qquad q = (\rho, \varphi, z)\ \sqrt{g} = \rho$$

$$\int d^3q\sqrt{g}\Psi_{n,\zeta}^+\Psi_{n'\zeta'} = \delta_{nn'}\delta_{\zeta\zeta'} + O(\hbar^{1/2}) \qquad \zeta = \pm 1\ n = (l, n_\rho, \nu).$$

For details about the construction of the Maslov canonical operator in the general case see [8–10] and [12,26] for the type of case considered here. We present only the final answer; explicit formulae for semiclassical wavefunctions (6.1), (6.2) for any $q \in \mathbb{R}^3_z$. Properties (i)–(iii) can be verified by direct calculations as in [26].

Let us consider the semiclassical wavefunctions (6.1) for the Schrödinger operator. We consider three regions (see figure 3). The first is called the 'shadow' region and is situated outside a neighbourhood of the annulus $R_- - \delta \leqslant \rho \leqslant R_+ + \delta, \delta > 0$. Here the wavefunction equals $O(\hbar^\infty), \hbar \to 0$. The second region is the interior of the annulus $R_- + \delta \leqslant \rho \leqslant R_+ - \delta$. We call it the bright region or domain of light. There the asymptotic wavefunctions have the form

$$\Psi = \Psi_{l,n_\rho,\nu}(\rho, \varphi, z, \hbar) = Ne^{il\varphi}\sum_{\pm}\Psi_\pm(\rho, z, \hbar)$$

$$= Ne^{il\hbar}\sum_{\pm}\frac{C_q^\pm}{\sqrt{\rho}}\left[\frac{e^{(i/\hbar)S(\tau_1)}}{(|dX(\tau_1)/d\tau_1|)^{1/2}}\Phi_\nu(\tau_1, z)\right]\Bigg|_{\substack{I=l\hbar \\ \tau_1=\tau_1^\pm(\rho) \\ E=E_{l,n_\rho,\nu}(\hbar)}} \tag{6.3}$$

where† $S(\tau_1)$ is the classical action along the closed curve $\bar{\Lambda}^1(I, E)$ (2.5) directed counterclockwise (see figure 2)

$$S(\tau_1) = \int_{\tau_1^0}^{\tau_1} P(\tau_1)\, dX(\tau_1). \tag{6.4}$$

The functions $\tau_1^\pm(\rho)$ are smooth (outside the turning points) branches of the equation

$$\rho = X(\tau_1) \qquad R_- + \delta \leqslant \rho \leqslant R_+ - \delta. \tag{6.5}$$

$$\Phi_\nu(\tau_1, z) = (\nu!)^{-1/2}(i/\sqrt{2})^\nu|Z(\tau_1)|^{-1/2}\exp\{-i(\nu + \tfrac{1}{2})\mathrm{Arg}Z(\tau_1)\}$$

$$\times \exp\{i\Omega(\tau_1)z^2/(2\hbar)\}H_\nu(z/(\sqrt{\hbar}|Z(\tau_1)|)) \qquad \nu = 0, 1, 2\ldots \tag{6.6}$$

where H_ν are the Hermite polynomials. The complex 'frequency' $\Omega(\tau_1)$ is determined by the stable solution $Z(\tau_1)$ of the Hill equation (2.18)

$$\Omega(\tau_1) = \frac{m\omega_1 Z'(\tau_1)}{Z(\tau_1)} \qquad \left(Z'(\tau_1) = \frac{dZ}{d\tau_1}\right) \qquad \mathrm{Im}\,\Omega(\tau_1) = \frac{1}{|Z(\tau_1)|^2} > 0. \tag{6.7}$$

$(\mathrm{Arg}Z(\tau_1))^{1/2}$ is the continuous branch of the square root of the argument of $Z(\tau_1) \neq 0$, $0 \leqslant \tau_1 \leqslant 2\pi$ at a fixed point τ_1^0. The constants $C_q^\pm$ equal $C_q^+ = 1$, $C_q^- = \exp(-i\pi/2)$. The

† For simplicity here and below we omit the dependence on the parameters I and E.

factors $\Phi_\nu(\tau_1, z)$ in (6.3) are the Fock states generated by the squeezed state $\Phi_{\nu=0}(\tau_1, z) = \Phi_0$ (see [27, 28]) of the non stationary (with respect to the 'intrinsic' time τ_1) harmonic oscillator with 2π-periodic variable frequency $\bar{\Omega}(\tau_1)$ (cf (2.16))

$$\begin{aligned} &-i\hbar\omega_1 \partial\Phi_0/\partial\tau_1 = (-(\hbar^2/2m)\partial^2/\partial z^2 + \bar{\Omega}(\tau_1)z^2/2)\Phi_0 \\ &\Phi_\nu(\tau_1, z) = (\nu!)^{-1/2}(\hat{a}^+)^\nu(\tau_1)\Phi_0(\tau_1, z) \end{aligned} \tag{6.8}$$

where $\hat{a}^+(\tau_1) = (2\hbar)^{-1/2}[m\omega_1 \bar{Z}'(\tau_1)(-i\hbar\partial/\partial z) + \bar{Z}(\tau_1)z]$ is a dynamical symmetry of equation (6.8).

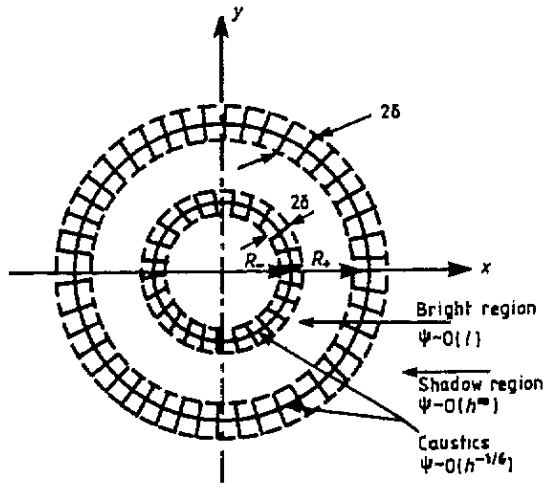


Figure 3. The structure of the support of individual 'superscarred' quantum states in the plane $z = 0$.

Finally, the third domain is the neighbourhood of the caustics $\rho = R_\pm$, $0 \leqslant \varphi < 2\pi$, $z = 0$. In the vicinity of the points $R_\pm = X(\tau_1^\pm)$, $dX/d\tau_1(\tau_1^\pm) = 0$, which are singular with respect to the projection of $\Lambda^2(I, E)$ on configuration space, we use (as in the standard WKB–Maslov method) the momentum representation by means of the Fourier transform $\mathcal{F}_{\rho\mapsto p_\rho}$ but only with respect to the radial coordinate ρ. Let us denote the semiclassical wavefunction here ($\rho \sim R_\pm$) as $\Psi^\pm$. Then

$$\begin{aligned} \Psi^\pm_{l,n_\rho,\nu}(\rho, \varphi, z, \hbar) &= N e^{il\varphi} C_p^\pm \mathcal{F}_{\rho\mapsto p_\rho}\left(\frac{\exp\{(i/\hbar)\tilde{S}(\tau_1)\}e_\pm(\tau_1)}{(|dP/d\tau_1(\tau_1)|)^{1/2}}\Phi_\nu(\tau_1, z)\Big|_{\tau_1=\tau_1^\pm(p_\rho)}\right) \\ &= \frac{N e^{il\varphi} C_p^\pm}{\sqrt{-2\pi i\hbar}} \int dp_\rho \left[\frac{e_\pm(\tau_1)\Phi_\nu(\tau_1, z)\exp\{(i/\hbar)(p_\rho\rho - \int_{\tau_1^0}^{\tau_1} X(\tau_1)\,dP(\tau_1))\}}{(|\omega_1^{-1}(\partial V_I/\partial\rho)(X(\tau_1), 0)|)^{1/2}}\right]\Bigg|_{\substack{I=l\hbar \\ \tau_1=\tilde{\tau}_1^\pm(p_\rho) \\ E=E_{l,n_\rho,\nu}(\hbar)}} \end{aligned} \tag{6.9}$$

$\tilde{\tau}_1^\pm(p_\rho)$ are the smooth branches of the solution of equation $p_\rho = P(\tau_1) = m\omega_1 X'(\tau_1)$, in a small neighbourhood where $|\tau_1 - \tau_1^\pm| < \varepsilon(\delta) \ll 1$: $\tau_1^\pm$: $P(\tau_1^\pm) = 0$ (see figure 2). The functions $e_\pm(\tau_1^\pm)$ are smooth and finite (cut-off functions): $e_\pm(\tau_1^\pm) = 1$ if $|\tau_1 - \tau_1^\pm| < \varepsilon$ but $e_\pm(\tau_1) = 0$ if $|\tau_1 - \tau_1^\pm| > \varepsilon$. The constants $C_p^\pm$ equal $C_p^+ = e^{-i\pi/2}C_p^- = e^{-i\pi}$.

In the formulae (6.3) and (6.9) N plays the role of a normalization constant: $N = (2T_1\pi^{3/2}\hbar^{1/2})^{-1/2}$, where T_1 is the period of radial oscillations (2.7).

Similar formulae are valid for the semiclassical wavefunctions *of the Pauli operator*. It follows from (6.2) that $\Psi_{n,\zeta} \sim v_\zeta = \text{const}$ and the scalar part of the asymptotic solution at

all points $q \in \mathbb{R}^3_q$ is determined by the next substitutions in formulae (6.3) and (6.9): ($g(\tau_1)$ was defined in (5.4))

$$S(\tau_1) \mapsto S(\tau_1) + \hbar\zeta g(\tau_1) \qquad \tilde{S}(\tau_1) \mapsto \tilde{S}(\tau_1) + \hbar\zeta g(\tau_1) \qquad \zeta = \pm 1.$$

Remark. The considered case $k_1 = k_2 = 0$ does not limit the generality of our considerations since, if we put $k_1 \neq 0$, $k_2 \neq 0$ in (5.5), then the series of semiclassical energy levels (up to $O(\hbar^2)$, $\hbar \to 0$) suffer only a renumeration of the orbital l and radial n_ρ quantum numbers:

$$E_{l,n_\rho,\nu,\zeta}(\hbar) + \hbar \sum_{j=i}^{2} \omega_j k_j = E_{l+k_2,n_\rho+k_1,\nu,\zeta}(\hbar) + O(\hbar^2)\ \hbar \to 0.$$

In this case the semiclassical eigenfunctions (up to $O(\hbar)$, $\hbar \to 0$) do not change:

$$\Psi_{l+k_2,n_\rho+k_1,\nu,\zeta} = \Psi_{l,n_\rho,\nu,\zeta} + O(\hbar) \qquad \hbar \to 0.$$

We have taken into account the fact that the actual expansion parameters in this case are the 'large' quantum numbers $|l| \gg 1$ and $n_\rho \gg 1$, which are related to the parameter $\hbar$, $\hbar \to 0$ by the conditions (4.2). In fact, selecting $k_2 \neq 0$, $k_1 \neq 0$, in formula (5.5) implies a change in the oscillating exponents of wavefunction (6.2): $e^{il\varphi} \mapsto e^{i(l+k_2)\varphi}$ and

$$\exp\left(\frac{i}{\hbar}S_\pm(\rho, E, I)\right)\Big|_{E=E_{l,n_\rho,\nu}} \mapsto \exp\left(\frac{i}{\hbar}S_\pm(\rho, E, I) + i\tau_1^\pm(\rho)\right)\Big|_{E=E_{l+k_2,n_\rho+k_1,\nu}}$$

respectively. Therefore, taking into consideration the fact that $|k_2| \ll l$, $|k_1| \ll n_\rho$ (as $\hbar \to 0$) we obtain the estimates

$$\begin{aligned}
e^{i(l+k_2)\varphi} &= e^{il\varphi}(1 + O(k_2/l))_{l\gg 1} \\
&= e^{il\varphi}(1 + O(\hbar))_{\hbar\to 0} \\
\exp\left(\frac{i}{\hbar}S_\pm(\rho, E) + i\tau_1^\pm(\rho)\right)\Big|_{E=E_{l+k_2,n_\rho+k_\rho,\nu},\, I=(l+k_2)\hbar} \\
&= e^{\frac{i}{\hbar}S_\pm(\rho,E,I)}|_{E=E_{l,n_\rho,\nu},\, I=l\hbar}(1 + O(k_2/l) + O(k_1/n_\rho))_{l\gg 1,\, n_\rho\gg 1} \\
&= e^{\frac{i}{\hbar}S_\pm(\rho,E_{l,n_\rho,\nu})}(1 + O(\hbar))\ \hbar \to 0.
\end{aligned}$$

The last estimates were obtained using a Taylor expansion in powers of k_2/l and k_1/n_ρ, together with the known formula $\partial S_\pm/\partial E(\rho, E, I) + \tau_1^\pm(\rho, E, I) = 0$ [29].

Analysis of formulae (6.1) and (6.2) show that the semiclassical wavefunctions $\Psi_{l,n_\rho,\nu}(\rho, \varphi, z, \hbar)$ are localized ($\hbar \to 0$), in a neighbourhood of the bright region filled by conditionally periodic trajectories (2.9). Inside the annulus, functions Ψ oscillate in the coordinates ρ and φ with frequency $\sim 1/\hbar$ and decrease exponentially in the coordinate z, similarly to Gaussian packets with the frequency of quantum oscillations of the order $\hbar^{-1/2}$, $\hbar \to 0$, determined by the zeros of the Hermite polynomials.

It is precisely in this sense that these semiclassical wavefunctions possess the *superscarring property* in the coordinate space, near the projection of the torus Λ^2. The functions (6.2) for the Pauli operator (1.1) have the same property.

Quantum superscarring of the wavefunction is non-uniform with respect to the classical motion along the coordinates ρ and z. The localization domain of probability density along z (near $z = 0$) is determined by dispersion $\sigma_z^\pm = \hbar(\operatorname{Im} Z(\tau_1^\pm(\rho)))^{-1}$ of the Gaussian packet $\Phi_{\nu=0}(\tau_1, z)$ (6.6): $|z| \leqslant \sqrt{\sigma_z^\pm} \sim \sqrt{\hbar}$, $\hbar \to 0$. Then inside the ring $R_- < \rho < R_+$: $|\Psi|^2(\rho, z) = O(1)$, but in the vicinity of the caustics $S^1_\pm$, i.e. as $\rho \sim R\pm$, the accumulation of density is *singular* as $\hbar \to 0$: $|\Psi|^2(R_\pm, z) \sim \hbar^{-1/3}$. This structure of the quantum

superscarring follows from the estimate for the integral in the right-hand part of (6.9); cf [26], where it is shown that, in a small neighbourhood of the turning points, $\rho - R_\pm \sim \hbar^{2/3}$, $\hbar \to 0$, the function (6.9) is the product of the standard Airy function with respect to ρ and of 'squeezed' states $\Phi_\nu(\tau_1, z)$, as $\tau = \tau_1^\pm(R_\pm)$.

For the model of external field considered it follows from [26] that, for $\rho \sim R_\pm$, the semiclassical wavefunctions $\Psi^\pm(\rho, \varphi, z)$ (6.9) have the form

$$\Psi^\pm(\rho, \varphi, z) = \frac{N e^{il\varphi} C_p^\pm}{\hbar^{1/6}} \sqrt{2\pi} e^{i\pi/4} D(\tau_1^\pm)/|\dot{P}(\tau_1^\pm)|^{-1/2} \times \exp\left\{\frac{i}{\hbar}\int_{\tau_1^0}^{\tau_1^\pm} P(\tau_1)\,dX(\tau_1) + P(\tau_1^\pm)(\rho - R_\pm)\right\} \mathrm{Ai}\left[D(\tau_1^\pm)\frac{\rho - R_\pm}{\hbar^{2/3}}\right] \times \Phi_\nu(\tau_1^\pm, z)$$

where the parameter $\tau_1^\pm$ determines the turning points $R_\pm = X(\tau_1^\pm)$, $X'(\tau_1^\pm) = 0$, $\mathrm{Ai}(x)$ is the standard Airy function, $\mathrm{Ai}(x) = (2\pi)^{-1}\int_{-\infty}^{\infty} \exp\{i(\xi x + \xi^3/3)\}\,d\xi$, $D(\tau_1) = \dot{P}(\tau_1)/m\omega_1\alpha(\tau_1)^{1/3}$ and $\alpha(\tau_1) = -(1/2m)(\partial V_l/\partial\rho(X(\tau_1), 0))^2$.

The qualitative behaviour of the probability density $|\Psi|^2(\rho, z)$ is displayed in figure 4, where the superscar structure of the semiclassical wavefunctions is illustrated.

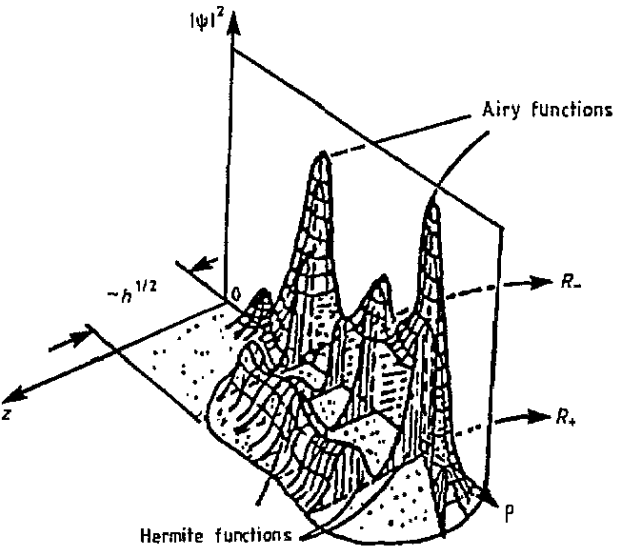


Figure 4. The qualitative behaviour of the density of 'superscarred' wavefunctions from (6.1); $R_\pm$ are the radial turning points.

7. Energy spectrum in the strong magnetic field approximation

Let us assume that the magnetic field is so strong that the Bohr electron radius $a_B = \hbar^2/me_0^2$ is much larger than the Larmour radius $a_H = (c\hbar/e_0H_0)^{1/2}$. The condition $a_H \ll a_B$ holds for values of the field starting with $H_0 \sim 4.7 \times 10^9$ G.

7.1. The Schrödinger operator

We obtain the asymptotic expansion† of the semiclassical energy spectrum $E_{l,n_\rho,\nu}(\hbar)$ in half-integer powers of $\varepsilon_S = a_H/a_B \ll 1$ from (4.1). The main difficulty is to calculate the Floquet index $\beta(I, E)$ for the stable solution of the Hill equation (2.16). We use the Poincaré–Lindstedt method to solve the Newton equation and the averaging method to calculate the Floquet index.

In the zero approximation as $\varepsilon_S \to 0$ (without taking into account the Coulomb interaction in the plane $z = 0$), in strong homogeneous magnetic fields, the radial variable ρ changes with high frequency $\omega_H = e_0 H_0/(mc) \to \infty$, $\omega_H/\omega_0 = 1/\varepsilon_S^2 > 1$, where ω_0 is the characteristic frequency of the classical motion in the Coulomb field. This allows us, at the second step in the calculation of $\beta(I, E)$, to apply the results of the averaging method in the theory of linear Hamiltonian systems with high frequency parametric perturbation (see [19]) to (2.16), and to obtain the expansion of $\beta(I, E)$ in powers of $\sqrt{\varepsilon_S}$, $\varepsilon_S \to 0$. Supplementing these computations with the standard expansion of the integral of motion $(2\pi\hbar^{-1} \oint_{\gamma_\rho} p\,dx)$ as $\varepsilon_S \to 0$, from quantization condition (4.1) we can obtain the approximate equation for the spectral parameter $E_{l,n_\rho,\nu}(\varepsilon_S, \hbar)$, with accuracy $O(\varepsilon_S^N)$, $\varepsilon_S \to 0$, $N = 1, 2, \ldots$. If we solve this equation by regular perturbation theory, we shall find the final representation of the semiclassical energy levels, corresponding to the family $\Lambda^2(I, E)$, for strong fields. The calculations according to this scheme are presented in the appendix.

Finally, in this approximation, the semiclassical energy spectrum $E_{l,n_\rho,\nu}(\hbar, \varepsilon_S)$ of the Schrödinger operator (1.7), associated with the two-dimensional quasiperiodic motion of the classical electron over the invariant isotropic tori $\Lambda^2(I, E)$ (2.6) can be expanded with respect to half-integer powers of the parameter $\varepsilon_S = a_H/a_B \ll 1$:

$$E_{l,n_\rho,\nu}(\hbar, \varepsilon_S) = \tfrac{1}{2}\omega_H\hbar(2n_\rho + |l| + l + \sqrt{\gamma}(1 + 2\nu) + \varepsilon_S^{1/2}\tilde{E}_1 + \varepsilon_S\tilde{E}_2 + O(\varepsilon^{3/2})) + O(\hbar^2). \tag{7.1}$$

$\tilde{E}_j$, $j = 0, 1, 2$ are shown in formulae (A.11) of the appendix.

7.2. The spin correction

To obtain the spin correction we calculate (see (5.4)), using the expansions (A.3)–(A.4), the integral

$$\int_0^{2\pi} X^{-3}(\tau_1, E, I, \varepsilon_S)\,d\tau_1$$

in powers of ε_S, $\varepsilon_S \to 0$. Up to $O(\varepsilon_S^2)$ we obtain

$$a_H^3 \int_0^{2\pi} X^{-3}(\tau_1, E, I, \varepsilon_S)\,d\tau_1 = 2\pi\alpha_1^2 + 2\alpha_1\alpha_2\varepsilon_S + O(\varepsilon_S^2),$$

where α_1 is defined in (A.8) and α_2 in (A.9).

Finally, the series of semiclassical energy levels of problem (1.1)–(1.2) corresponding to the family $\Lambda^2(I, E)$ (2.6) has the form

$$E_{l,n_\rho,\nu,\zeta}(\hbar) = E_{l,n_\rho,\nu}(\hbar, \varepsilon_S) + \hbar\zeta\left(\frac{e_0}{2mc}H_0 + \frac{e_0^2}{4m^2c^2}l\hbar\frac{\alpha_1(\tilde{E}_0)}{(a_H)^3}\right) \times \left[1 + \varepsilon_S\frac{\alpha_1(\tilde{E}_0)}{\pi\alpha_2(\tilde{E}_0)}\right]\Bigg|_{\tilde{E}_0 = 2n_\rho + |l| + (1+2\nu)\sqrt{\gamma}} + O(\varepsilon_S^2)) + O(\hbar^2) \tag{7.2}$$

† A similar expansion in ε_S^{-4} can be performed for weak fields. See [31] for $\gamma = 1$, $V_{s-f} = 0$.

$l = 0, \pm 1, \pm 2, \ldots, n_\rho = 0, 1, 2, \ldots, \nu = 0, 1, 2, \ldots, \zeta = \pm 1$ and $E_{l,n_\rho,\nu}(\hbar, \varepsilon_s)$ from (7.1).

8. Concluding remarks

(1) The spectral series constructed, related to the family of stable two-dimensional Lagrangian tori $\Lambda^2(I, E)$ with complex germ $r^3(\Lambda^2(I, E))$, describe the perturbed states of the system. These states are conditioned by the quick rotation of the electron ($|l| \gg 1$) and its radial oscillation of large amplitude ($n_\rho \gg 1$) in the plane $z = 0$. The quantum number $\nu = 0, 1, 2, \ldots$ determines the amplitude of the small axial (with respect to the field direction) oscillations: $\nu \ll n_\rho$, $\nu \ll |l|$. We emphasize that, in contrast to the semiclassical wavefunctions, the semiclassical energy levels $E_n(\hbar)$ (4.1), $E_{n,\zeta}(\hbar)$ (5.6), $n = (l, n_\rho, \nu)$, approximate the exact spectrum of problems (1.6) and (1.1) respectively, in the following sense. In a neighbourhood of order $\hbar^2$, $\hbar \to 0$ of the level $E_n(\hbar)$ of problem (1.6), there exists an exact value (maybe not unique) of the spectral parameter E_{exact}, such that

$$|E_{\text{exact}} - E_n(\hbar)| = O(\hbar^2) \qquad \hbar \to 0.$$

The same estimation is valid for the semiclassical energy levels $E_{n,\zeta}$, $\zeta = \pm 1$ of the Pauli operator. This fact follows from a well known general proposition (see [20, 30]) for the semiclassical asymptotics of a spectral problem when the asymptotic (as $\hbar \to 0$) wavefunction possesses properties (i)–(ii) (see section 6).

(2) Taking $\hbar n_\rho = O(\hbar)$, $\hbar \to 0$ (i.e. considering n_ρ to be small, $n_\rho \sim 1$) in (4.1), (5.6), (7.1), (7.2), (8.1), (8.2) and making the substitutions $n_\rho \to \nu_1$, $\nu \to \nu_2$, $\nu_1, \nu_2 = 0, 1, 2, \ldots$ we obtain the semiclassical energy spectrum corresponding to the motion of an electron along an equilibrium circle (see part I [13]).

(3) For the spectral problem (1.1)–(1.2), without taking into account the electron spin ($\hat{V}_{s-f} = 0$), formulae (4.1) and (5.6) for the semiclassical spectral series for the Schrödinger operator transform into the results of [31] if $\gamma = 1$, and of [32] for the anisotropic Kepler problem if $\gamma \neq 1$, $H_0 = 0$. As $\gamma = 1$, $H_0 = 0$, we obtain from (4.1) the exact spectrum of the hydrogen atom. Its main quantum number n equals $n_\rho + |l| + 2\nu + 1$. As $V_\gamma = 0$, $\varepsilon_s = 0$, we obtain from (7.1) the Landau levels of an electron in an homogeneous magnetic field, and the level number N equals $n_\rho + \nu + (l + |l|)/2$. Here the semiclassical quantum numbers n_ρ, l, ν enumerate the quantized two-dimensional Lagrangian tori.

(4) In the physical model of a non integrable system considered, we have concentrated our attention on the computational aspect of the semiclassical quantization by means of the complex WKB method. We did not touch some interesting problems related to the geometric–topological nature of the complex germ indices on isotropic manifolds (Lagrangian of less than full dimension). They are being intensively studied at present (e.g., [33–35]). In connection with this we note that the complex germ index $\hat{\beta}_{\gamma_\rho}$ (3.2), along the generatrix γ_ρ of the tori $\Lambda^2(I, E)$, can be separated into two parts: one topological and the other dynamical. its topological part is formed by the first term 2π, which can be more generally written as $\mu\pi$, where μ is the Maslov standard index† of the closed (Lagrangian) curve $\bar{\Lambda}^1(I, E)$ in the two-dimensional phase plane with coordinates (p_ρ, ρ). The dynamical part is related to the characteristic exponent $\beta(I, E)$ of the Floquet solution (2.17). Namely, its integer part $[\beta(I, E)]$ can be interpreted as the winding number of the closed phase trajectory $\bar{\Lambda}^1(I, E)$, with respect to the reduced four-dimensional phase space $\mathbb{R}^4_{p,q}$, with

† The complex constants $C_q^\pm$ and $C_p^\pm$ in (6.3) and (6.9) can be represented in the form $C_q^\pm = e^{i(\pi/2)\sigma_q^\pm}$ and $C_p^\pm = e^{-i(\pi/2)\sigma_p^\pm}$, where $\sigma_q^\pm$, $\sigma_p^\pm$ are the Maslov indices of charts of the canonical atlas on the curve $\bar{\Lambda}^1(I, E)$ (see [26] for details).

coordinates (p_ρ, p_z, ρ, z). This fact is general for stable closed trajectories of multi-dimensional Hamiltonian systems, when these trajectories are 'well enough' embedded into the phase space. This problem and the relationship between a complex germ index and the geometric properties of the Maslov index in the semiclassical Gutzwiller trace formula [29] have been widely studied (see, e.g., [36–39]) and will be considered in a future paper.

Second, let us note that the second quantization condition (3.5) of the family $[\Lambda^2(I, E), r^3(\Lambda^2(I, E))]$ along the cycle γ_ρ, has the same form as the semiclassical quantization formula of stable orbits that was obtained in [40], by expanding the amplitude of each trajectory into a geometric series in the Gutzwiller trace formula. A quantization condition similar to that in [40] is contained, as a particular case ($k = 1$), in the quantization conditions of Bohr–Sommerfeld type for families of k-dimensional isotropic manifolds Λ^k, with complex germs $r^n(\Lambda^k)$, from the complex WKB method given in [10]. Moreover, the quantization condition obtained there for stable closed phase curves includes the case when focal points are present†. It is worth noting also that the complex WKB method and its results (see, e.g., [10, 11], etc) are mathematically rigorously established.

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Appendix

Below we present the results of computation according to the scheme given above with accuracy $O(\varepsilon_S^2)$, $\varepsilon_S \to 0$. To do this it is convenient to pass to dimensionless variables: the coordinate x becomes $\rho = a_H x$ and the reduced spectral parameter $\tilde{E}$ becomes $E = \hbar\omega_H(\tilde{E} + l)/2$; I is equal to $l\hbar$, $|l| \gg 1$. In particular in this case equation (4.1) takes the following form

$$\int_{x_-(\varepsilon_S)}^{x_+(\varepsilon_S)} \left(\tilde{E} - \frac{x^2}{4} - \frac{l^2}{x^2} + \varepsilon_S \frac{2}{x}\right)^{1/2} \mathrm{d}x = \pi \left(n_\rho + \left(\frac{\beta(l\hbar, \tilde{E}, \varepsilon_S)}{2\pi} + 1\right)(\nu + \tfrac{1}{2})\right). \tag{A.1}$$

For the solution $X(t, \tilde{X}, l, \varepsilon_S)$ of equation (2.8), perturbation theory gives us the expansions of the amplitude and frequency of nonlinear oscillations in the form:

$$\begin{aligned} X(t, \tilde{E}, l, \varepsilon_S) &= X_0(\omega(\varepsilon_S)t, \tilde{E}, l) + \varepsilon_S X_1(\omega(\varepsilon_S)t, \tilde{E}, l) + O(\varepsilon_S^2) \\ \omega(\varepsilon_S) &= \omega_1^S + \varepsilon_S\omega_2^S + O(\varepsilon_S^2) \end{aligned} \tag{A.2}$$

where X_0 and X_1 are the following functions, 2π-periodic with respect to $\tau = \omega(\varepsilon_S)t$:

$$X_0(\tau, \tilde{E}, l) = \sqrt{2}(\tilde{E} + a\sin\tau)^{1/2} \qquad \tilde{E} \geqslant |l| \tag{A.3}$$

$$\begin{aligned} X_1(\tau, \tilde{E}, l) = &-2\frac{\partial X_0(\tau, \tilde{E}, l)}{\partial\tau}\int_0^\tau \frac{\partial X_0(\tau, \tilde{E}, l)}{\partial\tilde{E}} F(\tilde{\tau})\,\mathrm{d}\tilde{\tau} \\ &+2\frac{\partial X_0(\tau, \tilde{E}, l)}{\partial\tilde{E}}\int_0^\tau \frac{\partial X_0(\tau, \tilde{E}, l)}{\partial\tilde{\tau}} F(\tilde{\tau})\,\mathrm{d}\tilde{\tau} \qquad F(\tau) = -2\omega_1^\tau \frac{\partial^2 X_0}{\partial\tau^2} - X_0 - 2 \end{aligned} \tag{A.4}$$

$$\omega_1^S = \omega_H \qquad \omega_2^S = \omega_H\omega_1^\tau$$

† For closed geodesics without focal points on a compact Riemann manifold, similar quantization conditions were obtained in [41] for the first time.

$$\omega_1^\tau = \frac{\sqrt{2}}{2\pi a^2\sqrt{\tilde{E}+a}}\{\mathbb{E}(k(\tilde{E},l))(\tilde{E}+a) - \tilde{E}\mathbb{K}(k(\tilde{E},l))\} \tag{A.5}$$

where $\mathbb{E}(k)$ and $\mathbb{K}(k)$ are the complete elliptic integrals of the second and first kind respectively; $k = k(\tilde{E},l) = (2a/(\tilde{E}+a))^{1/2}$ and $a = (\tilde{E}^2 - l^2)^{1/2}$. Let us substitute these formulae into (2.16). We then find the expansion for the variable frequency $\Omega(t,\tilde{E},l,\varepsilon_S)$ in series of ε_S as $\varepsilon_S \to 0$

$$\Omega(t,\tilde{E},l,\varepsilon_S) = \Omega_0^S(\omega(\varepsilon_S)t,\tilde{E},l) + \varepsilon_S\Omega_1^S(\omega(\varepsilon_S)t,\tilde{E},l) + \mathrm{O}(\varepsilon_S^2) \tag{A.6}$$

where the frequencies Ω_0^S and m Ω_1^S are 2π-periodic in the first argument:

$$\Omega_0^S(\tau,\tilde{E},l) = \gamma e_0^2(mX_0^3(\tau,\tilde{E},l))^{-1}$$
$$\Omega_1^S(\tau,\tilde{E},l) = 3\gamma e_0^2X_1(\tau,\tilde{E},l)(mX_0^4(\tau,\tilde{E},l))^{-1}.$$

For $\omega_H \to \infty$, the averaging method for the Hill equation (2.16) with variable frequency (A.6) gives the following expansion for the Floquet exponents $\beta(I,\tilde{E})$:

$$\beta(I,\tilde{E})/2\pi = \sqrt{\gamma\varepsilon_S}\alpha_1(\tilde{E},l) + \varepsilon_S^{3/2}\sqrt{\gamma}\alpha_2(\tilde{E},l) + \mathrm{O}(\varepsilon_S^2) \tag{A.7}$$

$$\alpha_1(\tilde{E},l) = \left(\frac{1}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}\tau}{X_0^3(\tau,\tilde{E},l)}\right)^{1/2} = \left(\frac{\mathbb{E}(k(\tilde{E},l))}{\pi\sqrt{2}|l|\sqrt{\tilde{E}-a}}\right)^{1/2} \tag{A.8}$$

$$\alpha_2(\tilde{E},l) = -3(2\alpha_1(\tilde{E},l))^{-1}\int_0^{2\pi}X_1(\tau,\tilde{E},l)(X_0(\tau,\tilde{E},l))^4\,\mathrm{d}\tau. \tag{A.9}$$

For the integral on the left-hand side of (A.1) we find, to order $\mathrm{O}(\varepsilon_S^2)$, $\varepsilon_S \to 0$

$$J(\tilde{E},l,\varepsilon_S) = \int_{x_-(\varepsilon_S)}^{x_+(\varepsilon_S)}\left(\tilde{E} - \frac{x^2}{4} - \frac{l^2}{x^2} + \frac{2\varepsilon_S}{x}\right)^{1/2}\mathrm{d}x = J_0(\tilde{E},l) + \varepsilon_S J_1(\tilde{E},l) + \mathrm{O}(\varepsilon_S^2) \tag{A.10}$$

where

$$J_0(\tilde{E},l) = \pi(\tilde{E} - |l|)/2 \qquad J_1(\tilde{E},l) = \sqrt{2}\mathbb{K}(k)/\sqrt{\tilde{E}+a}.$$

From (A.1), (A.7)–(A.10), we obtain the following approximate (modulo $\mathrm{O}(\varepsilon_S^2)$) equation for the reduced energy $\tilde{E} = \tilde{E}_{l,n_\rho,\nu}(\hbar,\varepsilon_S)$

$$J_0(\tilde{E},l) + \varepsilon_S J_1(\tilde{E},l) = \pi[n_\rho + (\nu+\tfrac{1}{2})(1 + \sqrt{\gamma}(\varepsilon_S^{1/2}\alpha_1(\tilde{E},l) + \varepsilon_S^{3/2}\alpha_2(\tilde{E},l)))].$$

With the same accuracy $\mathrm{O}(\varepsilon_S^2)$, $\varepsilon_S \to 0$, its solution has the form

$$\tilde{E} = \tilde{E}_0 + \varepsilon_S^{1/2}\tilde{E}_1 + \varepsilon_S\tilde{E}_2 + \mathrm{O}(\varepsilon_S^{1/2})$$

$$\tilde{E}_0 = 2n_\rho + |l| + (1+2\nu)\sqrt{\gamma} \quad \tilde{E}_1 = 2\sqrt{\gamma}(\nu+\tfrac{1}{2})\left(\frac{\mathbb{E}(k(\tilde{E}_0))}{\pi\sqrt{2}|l|[\tilde{E}_0 - (\tilde{E}_0^2 - l^2)^{1/2}]^{1/2}}\right)^{1/2}$$

$$\tilde{E}_2 = -\frac{2\sqrt{2\gamma}\mathbb{K}(k)}{\pi[\tilde{E}_0 + (\tilde{E}_0^2 - l^2)^{1/2}]^{1/2}} + \frac{(\nu+1/2)\sqrt{\pi\gamma}}{\{\sqrt{2}|l|[\tilde{E}_0 - (\tilde{E}_0^2 - l^2)^{1/2}]\}^{1/2}(\tilde{E}_0^2 - l^2)}$$
$$\times\{\tilde{E}_0\cdot\mathbb{E}(k(\tilde{E}_0)) - [\tilde{E}_0 - (\tilde{E}_0^2 - l^2)^{1/2}]\mathbb{K}(k(\tilde{E}_0))\}. \tag{A.11}$$

References

[1] Schiff L I 1968 *Quantum Mechanics* 3rd edn (Tokyo: McGraw-Hill)
[2] Elliott R J and Loudon R 1960 *J. Phys. Chem. Sol.* **15** 146
[3] Herrick D R 1982 *Phys. Rev.* A **26** 323
[4] Turbiner A V 1980 *Zh. Eksp. Teor. Fiz.* **79** 868 (Engl. Transl. 1980 *Sov. Phys.–JETP* **52** 868)
[5] Turbiner A V 1984 *Usp. Fiz. Nauk* **144** 35 (in Russian) (Engl. Transl. 1984 *Sov. Phys.–Usp* **27** 668)
[6] Avron J, Herbst I and Simon B 1977 *Phys. Lett.* **62A** 214
[7] Arnol'd V I 1979 *Matematicheskie Metody Klassicheskoi Mexaniki* (Moscow: Nauka) (Engl. Transl. 1989 *Mathematical Methods of Classical Mechanics* (Berlin: Springer))
[8] Maslov V P 1973 *Operatornye Metody* (Moscow: Nauka) (in Russian) (Engl. Transl. 1976 *Operational Methods* (Moscow: Mir))
[9] Maslov V P 1977 *Kompleksnyi Metod WKB v Nelineinyx Uravnenyax* (Moscow: Nauka) (in Russian) (Engl. Transl. 1994 *The Complex WKB Method in Nonlinear Equations* (Basel: Birkhäuser))
[10] Belov V V and Dobrokhotov S Yu 1988 *Sov. Math. Dokl.* **37** 180
[11] Belov V V and Dobrokhotov S Yu 1988 *Preprint* (Steklov Mathematical Institute, Moscow) (in Russian)
[12] Belov V V and Dobrokhotov S Yu 1992 *Teor. Mat. Fiz.* **92** 215 (Engl. Transl. 1993 *Theor. Math. Phys.* **92** 843)
[13] Belov V V, Olivé V M and Volkova J L 1995 *J. Phys. A: Math. Gen.* **28** 5799
[14] Heller E J 1984 *Phys. Rev. Lett.* **53** 1515
[15] Bogomolny E 1988 *Physica* **31D** 169
[16] Berry M 1989 *Proc. R. Soc.* A **423** 219
[17] Robnik M 1989 *Preprint* Santa Barbara: Institute for Theoretical Physics, University of California
[18] Nayfeh A H and Mook D T 1979 *Nonlinear Oscillations* (New York: Wiley)
[19] Yakubovich V A and Starzhinskii V M 1975 *Linear Differential Equations with Periodic Coefficients* 2 vols (Melbourne: Krieger) (original Russian edn 1972)
[20] Maslov V P 1965 *Teorya Vozmuschenii i Asimptoticheskie Metody* (Moscow: Nauka) (in Russian) (French Transl. 1972 *Thèorie des Perturbations et Méthodes Asymptotiques* (Paris: Dunod))
[21] Keller J B and Rubinov S I 1963 *Phys. Rev.* **131** 131
[22] Leray J 1978 *Analyse Lagrangienne et Mécanique Quantique* (Strasbourg: College de France)
[23] Littlejohn R E and Flynn W G 1991 *Phys. Rev.* A **34** 5239
[24] Littlejohn R E and Flynn W G 1992 *Phys. Rev.* A **45** 7697
[25] Bagrov V G, Belov V V, Trifonov A Yu and Yevseyevich A A 1994 *J. Phys. A: Math. Gen.* **27** 1021
[26] Bagrov V G, Belov V V, Trifonov A Yu and Yevseyevich A A 1994 *J. Phys. A: Math. Gen.* **27** 5273
[27] Malkin I A and Man'ko V I 1981 *Dynamical Symmetries and Quantum Mechanics* (Dordrecht: Reidel)
[28] Perelomov A M 1987 *Generalized Coherent States and their Application* (Berlin: Springer)
[29] Gutzwiller M C 1990 *Chaos in Classical and Quantum Mechanics* (New York: Springer)
[30] Maslov V P and Fedoriuk M V 1976 *Kvaziklassicheskoe Priblizhenie v Kvantovoi Mexaniki* (Moscow: Nauka) (in Russian) (Engl. Transl. 1981 *Semiclassical Approximation in Quantum Mechanics* (Dordrecht: Reidel))
[31] Belov V V and Volkova J L 1993 *Russ. J. Math. Phys.* **1** 409
[32] Belov V V, Dobrokhotov S Yu and Olivé V M 1993 *Sov. Math. Dokl.* **331** 150
[33] Vorob'ev Yu M 1990 *Mat. Zametki* **48** 29
[34] Dobrokhotov S Yu and Olivé V M 1995 *Proc. Int. on Workshop Asymptotic Methods in Spectral Theory* (St Petersburg: Euler Int. Mathematical Institute (to be published by Gordon & Breach)
[35] Karasev M V and Vorob'ev Yu M 1992 *Preprint Amath-QDS-92-01* (Moscow Institute of Electronics and Mathematics) (Engl. Transl. 1995 *Adv. Math.*)
[36] Littlejohn R G and Robbins J M 1987 *Phys. Rev.* A **36** 2953
[37] Creagh S C, Robbins J M and Littlejohn R G 1990 *Phys. Rev.* A **42** 1907
[38] Robbins J M 1991 *Nonlinearity* **4** 343
[39] Eckhardt B, Fishman S, Müller K and Wintgen D 1992 *Phys. Rev.* A **45** 3531
[40] Voros A 1988 *J. Phys. A: Math. Gen.* **2** 685
[41] Babich V M 1968 *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov (LOMI)* **9** (in Russian)